

Resurgent functions and nonlinear systems of differential and difference equations

By

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Abstract

The principal aim of this article is to establish an iteration method on the space of resurgent functions. We discuss endless continuability of iterated convolution products of resurgent functions and derive their estimates developing the method in [KaS]. Using the estimates, we show the resurgence of formal series solutions of nonlinear differential and difference equations.

§ 1. Introduction

Resurgent analysis has its origin in the publications [E1] written by J. Écalle. It provides an effective method for the study of e.g. holomorphic dynamics, analytical differential equations, WKB analysis and it still fascinates many mathematicians and theoretical physicists. In this theory, the space of resurgent functions plays a central role: a formal series $\varphi(x) := \sum_{j=0}^{\infty} \varphi_j x^{-j} \in \mathbb{C}[[x^{-1}]]$ is resurgent if its formal Borel transform

$$\mathcal{B}(\varphi) := \varphi_0 \delta + \hat{\varphi}(\xi), \quad \hat{\varphi}(\xi) := \sum_{j=1}^{\infty} \varphi_j \frac{\xi^{j-1}}{(j-1)!}$$

is convergent and $\hat{\varphi}(\xi)$ is endlessly continuable (cf. [E1]). In this article, we adopt the definition of endless continuability in [CaNP]. As the Borel counterpart of Cauchy product in $\mathbb{C}[[x^{-1}]]$, the convolution product $\hat{\varphi} * \hat{\psi}$ of $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{C}\{\xi\}$ is defined as follows:

$$\hat{\varphi} * \hat{\psi}(\xi) := \int_0^{\xi} \hat{\varphi}(\xi - \xi') \hat{\psi}(\xi') d\xi'.$$

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To discuss analytic continuation of such a convolution product, the notion of *symmetrically contractible path* was introduced in [E1]. Following the principle in [CaNP], systematic construction of such paths was given in [S3] and detailed estimates for the convolution product of an arbitrary number of endlessly continuable functions were obtained in [S4] when the set of singular points of the functions is a closed discrete subset in \mathbb{C} and closed under addition (see also [MS]). Further, it was generalized in [OD] and [KaS] to the case where the location of singular points of endlessly continuable functions is written by a discrete filtered set (see Definition 2.1 for its definition). Especially in [KaS], a rigorous foundation for the analysis on the space of such endlessly continuable functions was provided: a structure of Fréchet space on the space of the functions was precisely given by the aid of endless Riemann surfaces (see Section 2). It allows us to handle analytical problems related to the convergence of the functions, e.g. substitution of resurgent functions to convergent series, implicit function theorem for resurgent functions.

However, we have still a problem in applying resurgent analysis to the study of analytical differential equations: there is no universally applicable way of proving the resurgence of formal series solutions of differential equations. Especially, it is important to determine the location of singular points of the Borel transformed formal series solutions for the use of alien calculus, which is the main tool of resurgent analysis.

Having these backgrounds in mind, we discuss the following question in this article: Can we extend the principle in [KaS] so that we can show the resurgence of formal series solutions of differential equations? The main purpose of this article is to establish an iteration method on the space of resurgent functions by developing the method in [KaS] and to show the resurgence of formal series solutions of differential equations by applying it. More precisely, we consider a nonlinear differential equation

$$(1.1) \quad \frac{d}{dx}\Phi = F(x^{-1}, \Phi)$$

at $x = \infty$ with $F(x^{-1}, \Phi) \in \mathbb{C}^n\{x^{-1}, \Phi\}$ satisfying the conditions $F(0, 0) = 0$ and $\det(\partial_\Phi F(0, 0)) \neq 0$. In this setting, (1.1) has a unique formal series solution $\Phi \in \mathbb{C}^n[[x^{-1}]]$. In [E1], Écalle claims each entry of Φ is resurgent. In [Co], Borel summability of transseries solutions of (1.1) was discussed and the singularity structure in the Borel plane of the solutions was precisely studied under non-resonance conditions. (See [Ku], [BKu1] and [BKu2] for the case of the difference equation (7.13).) However, our standpoint is close to Écalle's *mould calculus* rather than [Co]. Mould calculus was developed in [E1] and applied to the classification of saddle-node singularities in [E2] (see also [S1] and [S2]). It uses expansions by resurgent monomials associated with words generated by e.g. \mathbb{Z} and resurgent properties of formal integrals was studied by the use of the mould expansions. In this article, we use an expansion of $\hat{\Phi}$ by iterated

convolution products of meromorphic functions associated with *iteration diagrams* (see Definition 3.1) instead of words. Iterated convolution product is a combination of convolution product and Cauchy product determined by iteration diagrams (see Definition 3.6). Extending the estimates obtained in [KaS] to iterated convolution products of endlessly continuable functions, we obtain the following theorem as one of our main results:

Theorem 1.1. *The formal series solution $\Phi \in \mathbb{C}^n[[x^{-1}]]$ of (1.1) is resurgent.*

In Section 7, we describe detailed geometrical structure of singular points of $\hat{\Phi}$ by the use of discrete filtered set and reveal how the singular points are generated by the set of eigenvalues of $\partial_{\Phi}F(0,0)$.

The plan of this article is the following:

- Section 2 reviews the notions and the results related to Ω -resurgence.
- Section 3 introduces the notions of iteration diagram and iterated convolution. We give a key-estimate Theorem 3.9 for iterated convolution products of Ω -resurgent functions.
- Section 4 discusses the analytic continuation of iterated convolution products along a path γ using a (γ, T) -adapted deformation.
- Section 5 and Section 6 are devoted to the proof of Theorem 4.6: We construct a (γ, T) -adapted deformation $(\Psi_t)_{t \in [a,1]}$ in Section 5 and derive its estimates in Section 6.
- In Section 7, we show the resurgence of formal series solutions of nonlinear differential and difference equations using the estimate Theorem 3.9.

Some of the results in this article have been announced in [Ka].

§ 2. Preliminaries

In this section, we review the notions concerning Ω -resurgence of formal series discussed in [KaS].

Definition 2.1. We use the notation $\mathbb{R}_{\geq 0} = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$.

1. A *discrete filtered set*, or *d.f.s.* for short, is a family $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ of subsets of \mathbb{C} such that
 - a) Ω_L is a finite set,

- b) $\Omega_{L_1} \subseteq \Omega_{L_2}$ for $L_1 \leq L_2$,
 - c) there exists $\delta > 0$ such that $\Omega_\delta = \emptyset$.
2. Let Ω and Ω' be d.f.s. A d.f.s. $\Omega * \Omega'$ defined by the formula

$$(\Omega * \Omega')_L := \{ \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1}, \omega_2 \in \Omega'_{L_2}, L_1 + L_2 = L \} \cup \Omega_L \cup \Omega'_L \quad \text{for } L \in \mathbb{R}_{\geq 0}$$

is called the *sum* of d.f.s. Ω and Ω' . We set $\Omega^{*n} := \underbrace{\Omega * \cdots * \Omega}_{n \text{ times}}$ for $n \geq 1$ and define a d.f.s. $\Omega^{*\infty}$ by

$$\Omega^{*\infty} := \varinjlim_n \Omega^{*n}.$$

3. A *trivial* d.f.s. $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ is a d.f.s. satisfying $\Omega_L = \emptyset$ for all $L \in \mathbb{R}_{\geq 0}$ and we denote it by \emptyset .
4. Given a d.f.s. Ω , the *distance* to Ω is the number $\rho(\Omega) := \sup\{ \rho \in \mathbb{R}_{\geq 0} \mid \Omega_\rho = \emptyset \}$.

We define for a d.f.s. Ω

$$\mathcal{S}_\Omega := \{ (\lambda, \omega) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \omega \in \Omega_\lambda \},$$

$$\mathcal{M}_\Omega := (\mathbb{R}_{\geq 0} \times \mathbb{C}) \setminus \overline{\mathcal{S}_\Omega},$$

where $\overline{\mathcal{S}_\Omega}$ denotes the closure of \mathcal{S}_Ω in $\mathbb{R}_{\geq 0} \times \mathbb{C}$.

Let Π be the set of all Lipschitz paths $\gamma : [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(0) = 0$. We denote the restriction of $\gamma \in \Pi$ to the interval $[0, t]$ for $t \in [0, 1]$ by $\gamma|_t$ and the total length of $\gamma|_t$ by $L(\gamma|_t)$.

Definition 2.2. Given a d.f.s. Ω , we call $\gamma \in \Pi$ *Ω -allowed path* if it satisfies

$$\tilde{\gamma}(t) := (L(\gamma|_t), \gamma(t)) \in \mathcal{M}_\Omega \quad \text{for all } t \in [0, 1],$$

and denote the set of all Ω -allowed paths by Π_Ω .

Remark 2.3. When a piecewise C^1 path $t \in [0, 1] \mapsto \tilde{\gamma}(t) = (\lambda(t), \gamma(t)) \in \mathcal{M}_\Omega$ with $\tilde{\gamma}(0) = (0, 0)$ is given, the Ω -allowedness of γ is characterized by the condition $\lambda'(t) = |\gamma'(t)|$ for a.e. $t \in [0, 1]$.

Recall that an Ω -endless Riemann surface is a triple $(X, \mathfrak{p}, \underline{0})$ such that X is a connected Riemann surface, $\mathfrak{p} : X \rightarrow \mathbb{C}$ is a local biholomorphism, $\underline{0} \in \mathfrak{p}(0)$, and any path $\gamma : [0, 1] \rightarrow \mathbb{C}$ of Π_Ω has a lift $\underline{\gamma} : [0, 1] \rightarrow X$ such that $\underline{\gamma}(0) = \underline{0}$. A

morphism $\mathfrak{q} : (X, \mathfrak{p}, \underline{0}) \rightarrow (X', \mathfrak{p}', \underline{0}')$ of Ω -endless Riemann surfaces is given by a local biholomorphism $\mathfrak{q} : X \rightarrow X'$ such that the following diagram is commutative:

$$\begin{array}{ccc} (X, \underline{0}) & \xrightarrow{\mathfrak{q}} & (X', \underline{0}') \\ & \searrow \mathfrak{p} \quad \swarrow \mathfrak{p}' & \\ & (\mathbb{C}, 0) & \end{array}$$

The existence of the initial object $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ in the category of Ω -endless Riemann surfaces was proved in [KaS]:

Theorem 2.4 ([KaS]). *There exists an Ω -endless Riemann surface $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ such that X_Ω is simply connected and, for any Ω -endless Riemann surface $(X, \mathfrak{p}, \underline{0})$, there is a unique morphism*

$$\mathfrak{q} : (X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega) \rightarrow (X, \mathfrak{p}, \underline{0}).$$

Let $\hat{\mathcal{R}}_\Omega$ denote the space of Ω -continuable functions, i.e., holomorphic germs $\hat{\varphi} \in \mathbb{C}\{\xi\}$ which can be analytically continued along any path $\gamma \in \Pi_\Omega$. Then, there exists an isomorphism

$$\mathfrak{p}_\Omega^* : \hat{\mathcal{R}}_\Omega \xrightarrow{\sim} \Gamma(X_\Omega, \mathcal{O}_{X_\Omega}),$$

where \mathcal{O}_{X_Ω} is the sheaf of holomorphic functions on X_Ω , and hence, a structure of Fréchet space is naturally introduced to $\hat{\mathcal{R}}_\Omega$ as follows: We set for $L, \delta > 0$

$$\mathcal{M}_\Omega^{\delta, L} := \{ (\lambda, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \text{dist}((\lambda, \xi), \overline{\mathcal{S}}_\Omega) \geq \delta, \lambda \leq L \},$$

$$\Pi_\Omega^{\delta, L} := \{ \gamma \in \Pi_\Omega \mid (L(\gamma|_t), \gamma(t)) \in \mathcal{M}_\Omega^{\delta, L} \text{ for all } t \in [0, 1] \},$$

where $\text{dist}(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$, and define compact subsets $K_\Omega^{\delta, L}$ of X_Ω by

$$K_\Omega^{\delta, L} := \{ \gamma(1) \in X_\Omega \mid \gamma \in \Pi_\Omega^{\delta, L} \}.$$

Since X_Ω is exhausted by $(K_\Omega^{\delta, L})_{\delta, L > 0}$, a family of seminorms $\| \cdot \|_\Omega^{\delta, L}$ ($\delta, L > 0$) defined by

$$\| \hat{\varphi} \|_\Omega^{\delta, L} := \sup_{\xi \in K_\Omega^{\delta, L}} | \mathfrak{p}_\Omega^* \hat{\varphi}(\xi) | \quad \text{for } \hat{\varphi} \in \hat{\mathcal{R}}_\Omega$$

induces a structure of Fréchet space on $\hat{\mathcal{R}}_\Omega$. Correspondingly, a family of seminorms $\| \cdot \|_\Omega^{\delta, L}$ ($\delta, L > 0$) on the space of Ω -resurgent series

$$\mathcal{R}_\Omega := \mathcal{B}^{-1}(\mathbb{C}\delta \oplus \hat{\mathcal{R}}_\Omega)$$

are defined by

$$\| \varphi \|_\Omega^{\delta, L} := | \varphi_0 | + \| \hat{\varphi} \|_\Omega^{\delta, L} \quad \text{for } \varphi \in \mathcal{R}_\Omega,$$

where $\mathcal{B}(\varphi) = \varphi_0 \delta + \hat{\varphi} \in \mathbb{C}\delta \oplus \hat{\mathcal{R}}_\Omega$.

Remark 2.5. Notice that $(X_\emptyset, \mathfrak{p}_\emptyset, \underline{Q}_\emptyset) \xrightarrow{\sim} (\mathbb{C}, \text{id}_\mathbb{C}, 0)$, and hence, $\hat{\mathcal{R}}_\emptyset \xrightarrow{\sim} \Gamma(\mathbb{C}; \mathcal{O}_\mathbb{C})$. Since $K_\emptyset^{\delta, L} = \{\xi \in \mathbb{C} \mid |\xi| \leq L\}$ for $\delta, L > 0$, we have $\|\hat{\varphi}\|_\emptyset^{\delta, L} = \sup_{|\xi| \leq L} |\hat{\varphi}(\xi)|$ for $\hat{\varphi} \in \hat{\mathcal{R}}_\emptyset$.

Now, let Ω' be a d.f.s. satisfying $\Omega \subset \Omega'$. From Theorem 2.4, we find that there exists a morphism $\mathfrak{q} : (X_{\Omega'}, \mathfrak{p}_{\Omega'}, \underline{Q}_{\Omega'}) \rightarrow (X_\Omega, \mathfrak{p}_\Omega, \underline{Q}_\Omega)$. Since $\mathfrak{q}(K_{\Omega'}^{\delta, L}) \subset K_\Omega^{\delta, L}$, we have

$$\|\hat{\varphi}\|_{\Omega'}^{\delta, L} \leq \|\hat{\varphi}\|_\Omega^{\delta, L} \quad \text{for } \hat{\varphi} \in \hat{\mathcal{R}}_\Omega,$$

and hence,

$$\|\varphi\|_{\Omega'}^{\delta, L} \leq \|\varphi\|_\Omega^{\delta, L} \quad \text{for } \varphi \in \mathcal{R}_\Omega.$$

§ 3. Iterated convolution of resurgent functions

§ 3.1. Iteration diagram

Definition 3.1. Let $T = (V, E)$ be a directed tree diagram, where V (resp. E) is the set of vertices (resp. edges) of T . We call T *iteration diagram* if T satisfies the condition that any vertex $v \in V$ has at most one outgoing edge. We denote the set of iteration diagrams by \mathcal{T} .

Since $T \in \mathcal{T}$ is connected and has no cycles, we immediately have the following

Lemma 3.2. *Each $T \in \mathcal{T}$ has a unique vertex \hat{v} such that there exists a path $v \rightarrow \cdots \rightarrow \hat{v}$ from v to \hat{v} in T for any vertex $v \in V$ and such a path is unique.*

Definition 3.3. Let $T = (V, E)$ be an iteration diagram.

1. We call \hat{v} in Lemma 3.2 *root* of T .
2. We call a vertex v *leaf* of T if v has no edge e such that the terminal vertex of e is v and denote the set of leaves of T by L .
3. The *branch* $T_v = (V_v, E_v)$ of T at $v \in V$ is the diagram that consists of the vertexes $u \in V$ that have a path $u \rightarrow \cdots \rightarrow v$ from u to v in T and the edges $v_1 \xrightarrow{e} v_2 \in E$ such that $v_1, v_2 \in V_v$.

From the definition of the branch, we obtain the following

Lemma 3.4. *Given $T \in \mathcal{T}$, the branch T_v of T at each vertex $v \in V$ defines an iteration diagram with the root v .*

Notation 3.5. Let $T = (V, E)$ be an iteration diagram.

1. We set $V^\circ := V \setminus \{\hat{v}\}$.
2. For each $v \in V^\circ$, there exists a unique vertex u that has an edge $v \rightarrow u$. We denote such a vertex by v_\uparrow .
3. Given $v \in V$, we denote the set of vertices $u \in V$ that have an edge $u \rightarrow v$ by V_v^1 .

We assign each vertex v a weight w_v defined as the cardinal of $\{v' \in L \mid \exists \text{ a path } v' \rightarrow \dots \rightarrow v\}$. Notice that w_v satisfies

$$\begin{cases} w_v = 1 & (v \in L), \\ w_v = \sum_{u \in V_v^1} w_u & (v \in V \setminus L). \end{cases}$$

Iteration diagrams are graded by the cardinal $|V|$ of vertexes:

$$\mathcal{T} = \bigsqcup_{k=1}^{\infty} \mathcal{T}_k, \quad \mathcal{T}_k = \{T = (V, E) \in \mathcal{T} \mid |V| = k\}.$$

§ 3.2. Iterated convolution

Let $T = (V, E) \in \mathcal{T}_k$ ($k \geq 1$) be an iteration diagram and assume that analytic germs $\hat{f}_v, \hat{\varphi}_v \in \mathbb{C}\{\xi\}$ are assigned to each vertex $v \in V$. Starting from the leaves of T , we inductively construct $\{\hat{\psi}_v\}_{v \in V}$ from $\{\hat{f}_v\}_{v \in V}$ and $\{\hat{\varphi}_v\}_{v \in V}$ by the rule

$$(3.1) \quad \hat{\psi}_v := \hat{\varphi}_v \cdot \left(\hat{f}_v * \prod_{u \in V_v^1}^* \hat{\psi}_u \right) \quad (v \in V),$$

where $\prod_{u \in V_v^1}^* \hat{\psi}_u$ is the convolution product of $\hat{\psi}_u$ over all the vertices $u \in V_v^1$ and we regard it as the unit δ when $v \in L$.

Definition 3.6. Given $T \in \mathcal{T}$ and $\{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V} \subset \mathbb{C}\{\xi\}$, we call $\hat{\psi}_T := \hat{\psi}_{\hat{v}}$ defined by the rule (3.1) *iterated convolution* of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$.

Notation 3.7. For an iteration diagram $T = (V, E) \in \mathcal{T}_k$ ($k \geq 1$), we set

$$\Delta_T := \left\{ (s_v)_{v \in V} \in \mathbb{R}_{\geq 0}^k \mid \sum_{u \in V_v^1} s_u \leq s_v, s_{\hat{v}} = 1 \right\}$$

and $[\Delta_T] \in \mathcal{E}_{k-1}(\mathbb{R}^k)$ denotes the corresponding integration current, where the orientation of Δ_T is defined so that it satisfies

$$(3.2) \quad \int_{\Delta_T} \bigwedge_{v \in V^\circ} ds_v = \frac{1}{(k-1)!}.$$

We set for $\rho > 0$

$$D_\rho := \{ \xi \in \mathbb{C} \mid |\xi| < \rho \}.$$

Let Ω be a d.f.s. We define a map $\mathcal{L}_v : D_{\rho(\Omega)} \rightarrow X_{\Omega_v}$ by

$$\mathcal{L}_v(\xi) := \underline{\gamma}_\xi(1) \quad \text{for } \xi \in D_{\rho(\Omega)},$$

where $\Omega_v := \Omega^{*w_v}$ ($v \in V$) and $\underline{\gamma}_\xi : [0, 1] \rightarrow X_{\Omega_v}$ is the lift of the path $\gamma_\xi : t \in [0, 1] \mapsto t\xi$. Notice that \mathcal{L}_v gives a local isomorphism from $D_{\rho(\Omega)}$ to an open neighborhood $\mathcal{L}_v(D_{\rho(\Omega)})$ of $\underline{\Omega}_v \in X_{\Omega_v}$.

Now, assume that $\{\hat{f}_v\}_{v \in V} \subset \hat{\mathcal{R}}_\emptyset$ and $\hat{\varphi}_v \in \hat{\mathcal{R}}_{\Omega_v}$ for $v \in V$. We consider a map

$$\mathcal{D}(\xi) : \vec{s} = (s_v)_{v \in V} \mapsto \mathcal{D}(\xi, \vec{s}) := (\mathcal{L}_v(s_v \xi))_{v \in V} \in X_\Omega^T \quad \text{for } \xi \in D_{\rho(\Omega)}$$

defined on a neighborhood of Δ_T in \mathbb{R}^k , where

$$X_\Omega^T := \prod_{v \in V} X_{\Omega_v}.$$

Let $\mathcal{D}(\xi)_\#[\Delta_T] \in \mathcal{E}_{k-1}(X_\Omega^T)$ denote the push-forward of $[\Delta_T]$ by $\mathcal{D}(\xi)$. Then, we have the following representation of the iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$:

Proposition 3.8. *Given $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$, define a holomorphic $(k-1)$ -form β_T on X_Ω^T by*

$$\beta_T := \left(\prod_{v \in V} (\mathfrak{p}_{\Omega_v}^* \hat{\varphi}_v)(\underline{\xi}_v) \hat{f}_v(\xi_v - \sum_{u \in V_v^1} \xi_u) \right) \bigwedge_{v \in V^\circ} d\underline{\xi}_v,$$

where $\bigwedge_{v \in V^\circ} d\underline{\xi}_v$ is the pullback of the $(k-1)$ -form $\bigwedge_{v \in V^\circ} d\xi_v$ in X_Ω^T by $(\mathfrak{p}_{\Omega_v})_{v \in V} : X_\Omega^T \rightarrow \mathbb{C}^k$ and $\xi_v = \mathfrak{p}_{\Omega_v}(\underline{\xi}_v)$ ($v \in V$). Then, the following equality holds for $\xi \in D_{\rho(\Omega)}$:

$$(3.3) \quad \hat{\psi}_T(\xi) = \mathcal{D}(\xi)_\#[\Delta_T](\beta_T).$$

Proof. We prove (3.3) by induction. We first note that $\mathcal{D}(\xi)_\#[\Delta_T](\beta_T)$ is regarded as $(\mathfrak{p}_{\Omega}^* \hat{\varphi}_{\hat{v}})(\underline{\xi}_{\hat{v}}) \hat{f}_{\hat{v}}(\xi_{\hat{v}})|_{\underline{\xi}_{\hat{v}} = \mathcal{D}(\xi)} = \hat{\varphi}_{\hat{v}}(\xi) \hat{f}_{\hat{v}}(\xi)$ when $T \in \mathcal{T}_1$, and hence, the equality (3.3) holds for the diagram T_v ($v \in L$). Next, take $v \in V$ and assume that (3.3) holds for all the branches T_u ($u \in V_v^\circ$). From the definition of the iterated convolution (3.1), we have the following representation of $\hat{\psi}_{T_v}$:

$$\hat{\psi}_{T_v}(\xi) = \hat{\varphi}_v(\xi) \int_{\Delta_\ell} \hat{f}_v(\xi(1 - \sum_{u \in V_v^1} s_u)) \prod_{u \in V_v^1} \hat{\psi}_{T_u}(\xi s_u) \bigwedge_{u \in V_v^1} \xi ds_u,$$

where $\ell = |V_v^1|$ and Δ_ℓ is the ℓ -dimensional simplex defined by

$$\Delta_\ell := \{ (s_u)_{u \in V_v^1} \in \mathbb{R}_{\geq 0}^\ell \mid \sum_{u \in V_v^1} s_u \leq 1 \}.$$

Since Δ_{T_v} is rewritten as

$$\left\{ (s_{\tilde{v}})_{\tilde{v} \in V_v} \left| \begin{array}{l} s_{\tilde{v}} = s_u \tilde{s}_{\tilde{v}}, (\tilde{s}_{\tilde{v}}) \in \Delta_{T_u}, (s_u) \in \Delta_\ell \\ \text{for } \tilde{v} \in V_u \text{ and } u \in V_v^1, s_v = 1 \end{array} \right. \right\},$$

we obtain (3.3) for the diagram T_v from the induction hypothesis. It proves (3.3) for $T = T_{\hat{v}}$. \square

We now state one of our main theorems:

Theorem 3.9. *Let Ω be a d.f.s. and let $\delta, L > 0$ be reals such that $2\delta < \rho(\Omega)$. Then, there exist $c, \delta' > 0$ such that, for every $T = (V, E) \in \mathcal{T}_k$ ($k \geq 1$), $\{\hat{f}_v\}_{v \in V} \subset \hat{\mathcal{R}}_\emptyset$ and $\hat{\varphi}_v \in \hat{\mathcal{R}}_{\Omega_v}$ ($v \in V$), the iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ is $\Omega_{\hat{v}}$ -continuable and satisfies the following estimates:*

$$(3.4) \quad \|\hat{\psi}_T\|_{\Omega_{\hat{v}}}^{\delta, L} \leq \frac{c^{k-1}}{(k-1)!} \sup_{\vec{s} \in \Delta_T} \prod_{v \in V} \|\hat{\varphi}_v\|_{\Omega_v}^{\delta', s_v L} \|\hat{f}_v\|_{\emptyset}^{\delta', s_v L}.$$

Since $\Omega \subset \Omega_v$ for all $v \in V$, we obtain the following

Corollary 3.10. *Under the same assumptions with Theorem 3.9, there exist $c, \delta' > 0$ such that, for every $T = (V, E) \in \mathcal{T}_k$ ($k \geq 1$), $\{\hat{f}_v\}_{v \in V} \subset \hat{\mathcal{R}}_\emptyset$ and $\{\hat{\varphi}_v\}_{v \in V} \subset \hat{\mathcal{R}}_\Omega$, the iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ satisfies*

$$(3.5) \quad \|\hat{\psi}_T\|_{\Omega_{\hat{v}}}^{\delta, L} \leq \frac{c^{k-1}}{(k-1)!} \prod_{v \in V} \|\hat{\varphi}_v\|_{\Omega}^{\delta', L} \|\hat{f}_v\|_{\emptyset}^{\delta', L}.$$

The proof of Theorem 3.9 will be given in Section 4.

§ 4. (γ, T) -adapted deformation of $\mathcal{D}(\gamma(a))$

In this section, we introduce the notion of (γ, T) -adapted deformation of $\mathcal{D}(\gamma(a))$, which is a slight generalization of γ -adapted origin-fixing isotopies in [S4, Def. 5.1]. Let $T = (V, E)$ be an iteration diagram and let Ω be a d.f.s. We take $\rho > 0$ such that $2\rho < \rho(\Omega)$. We fix a path $\gamma : [0, 1] \rightarrow \mathbb{C}$ in $\Pi_{\Omega_{\hat{v}}}^{\delta, L}$ with $L > 0$ and $\delta \in (0, \rho]$ satisfying the following condition:

$$(4.1) \quad \text{there exists } a \in (0, 1) \text{ such that } \gamma|_{[a, 1]} \text{ is } C^1, |\gamma(a)| = \rho \text{ and } \gamma(t) = \gamma(a)a^{-1}t \text{ for } t \in [0, a].$$

Notation 4.1. Given $T = (V, E) \in \mathcal{T}$, we set

$$\begin{aligned}\Delta_{T,v}^0 &:= \{(s_u)_{u \in V} \in \Delta_T \mid s_v = 0\}, \\ \Delta_{T,v}^1 &:= \{(s_u)_{u \in V} \in \Delta_T \mid \sum_{u \in V_v^1} s_u = s_v\}, \\ \mathcal{N}_v^0 &:= \{(\underline{\xi}_u)_{u \in V} \in X_\Omega^T \mid \underline{\xi}_v = \underline{0}_{\Omega_v}\}, \\ \mathcal{N}_v^1 &:= \{(\underline{\xi}_u)_{u \in V} \in X_\Omega^T \mid \sum_{u \in V_v^1} \mathfrak{p}_{\Omega_u}(\underline{\xi}_u) = \mathfrak{p}_{\Omega_v}(\underline{\xi}_v)\},\end{aligned}$$

where $\Delta_{T,v}^0$ and \mathcal{N}_v^0 (resp. $\Delta_{T,v}^1$ and \mathcal{N}_v^1) are defined for $v \in V^\circ$ (resp. $v \in V \setminus L$).

Definition 4.2. We call a family of maps $\Psi_t : \Delta_T \rightarrow X_\Omega^T$ ($t \in [a, 1]$) (γ, T) -adapted deformation of $\mathcal{D}(\gamma(a))$ in X_Ω^T if it satisfies the following conditions:

1. $\Psi_a = \mathcal{D}(\gamma(a))$,
2. the map $(t, \vec{s}) \in [a, 1] \times \Delta_T \mapsto \Psi_t(\vec{s}) \in X_\Omega^T$ is locally Lipschitz,
3. the v -th component $\underline{\xi}_v^t$ of $\vec{\xi}^t := \Psi_t(\vec{s})$ depends only on the variables s_u ($u \in W_v$), where

$$W_v := \{\hat{v}\} \cup \bigcup_{v'} V_{v'}^1$$

and the union $\bigcup_{v'}$ is taken over all the vertexes v' on the path from v_\uparrow to \hat{v} ,

4. $\underline{\xi}_{\hat{v}}^t(\vec{s}) = \underline{\gamma}(t)$ holds for any $t \in [a, 1]$ and $\vec{s} \in \Delta_T$,
5. $\Psi_t(\Delta_{T,v}^0) \subset \mathcal{N}_v^0$ holds for any $t \in [a, 1]$ and $v \in V^\circ$,
6. $\Psi_t(\Delta_{T,v}^1) \subset \mathcal{N}_v^1$ holds for any $t \in [a, 1]$ and $v \in V \setminus L$.

We now show the following

Proposition 4.3. Consider an iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ with the data $T = (V, E) \in \mathcal{T}$, $\{\hat{f}_v\}_{v \in V} \subset \hat{\mathcal{R}}_\emptyset$ and $\hat{\varphi}_v \in \hat{\mathcal{R}}_{\Omega_v}$ ($v \in V$) and assume that a (γ, T) -adapted deformation $(\Psi_t)_{t \in [a, 1]}$ of $\mathcal{D}(\gamma(a))$ is given. Then, the analytic continuation of $\mathfrak{p}_{\Omega_{\hat{v}}}^* \hat{\psi}_T$ along $\underline{\gamma}$ is written as follows:

$$(\mathfrak{p}_{\Omega_{\hat{v}}}^* \hat{\psi}_T)(\underline{\gamma}(t)) = (\Psi_t)_\# [\Delta_T](\beta_T) \quad \text{for } t \in [a, 1].$$

Notice that the situation discussed in [S4] and [KaS] is regarded as the case where the iteration diagram $T \in \mathcal{T}_k$ satisfies $V_{\hat{v}}^1 = V^\circ$ with $\hat{f}_{\hat{v}} = \hat{\varphi}_{\hat{v}} = 1$. We first note the following

Lemma 4.4 ([S4]). *Let $T = (V, E)$ be an iteration diagram such that $V_{\hat{v}}^1 = V^\circ$ and assume that a (γ, T) -adapted deformation $(\Psi_t)_{t \in [a, 1]}$ of $\mathcal{D}(\gamma(a))$ is given. Then, the analytic continuation of $\mathfrak{p}_{\Omega_{\hat{v}}}^* \hat{\psi}_T$ along $\underline{\gamma}$ is written as follows:*

$$(\mathfrak{p}_{\Omega_{\hat{v}}}^* \hat{\psi}_T)(\underline{\gamma}(t)) = (\Psi_t)_\# [\Delta_T](\beta_T) \quad \text{for } t \in [a, 1].$$

Indeed, in this case, we can regard β_T as a holomorphic $(k-1)$ -form on X_Ω^{k-1} with a holomorphic parameter $\underline{\xi}_{\hat{v}} \in X_{\Omega^{*(k-1)}}$. Therefore, adapting the proof of [S4, Prop. 5.4] and restricting the parameter $\underline{\xi}_{\hat{v}}$ to $\underline{\gamma}(t)$, we obtain Lemma 4.4.

Notation 4.5. Let a (γ, T) -adapted deformation $(\Psi_t)_{t \in [a, 1]}$ of $\mathcal{D}(\gamma(a))$ be given and assume that $(s_u)_{u \in V \setminus V_v^\circ}$ is taken so that it satisfies

$$(4.2) \quad \left\{ ((s_u)_{u \in V \setminus V_v}, (s_v \tilde{s}_u)_{u \in V_v}) \mid (\tilde{s}_u)_{u \in V_v} \in \Delta_{T_v} \right\} \subset \Delta_T.$$

Let $\text{pr}_{T_v} : X_\Omega^T \rightarrow X_\Omega^{T_v}$ be the natural projection. We define a map

$$\Psi_t|_{T_v}((s_u)_{u \in V \setminus V_v^\circ}; \cdot) : \Delta_{T_v} \rightarrow X_\Omega^{T_v}$$

by

$$\Psi_t|_{T_v}((s_u)_{u \in V \setminus V_v^\circ}; \vec{s}') := \text{pr}_{T_v} \circ \Psi_t((s_u)_{u \in V \setminus V_v}, (s_v s_u)_{u \in V_v})$$

for $\vec{s}' := (s_u)_{u \in V_v} \in \Delta_{T_v}$. We note that the map $\Psi_t|_{T_v}$ depends only on the parameters s_u ($u \in W_v$), and hence, we write the map as $\Psi_t|_{T_v}((s_u)_{u \in W_v}; \cdot)$. (See Definition 4.2.3.)

Since the path $\underline{\gamma}_v((s_u)_{u \in W_v}; \cdot) : [0, 1] \rightarrow X_{\Omega_v}$ obtained by concatenating the paths $t \in [0, a] \mapsto \mathcal{L}_v(\gamma(a)s_v t/a)$ and the v -th component of the path $t \in [a, 1] \mapsto \Psi_t|_{T_v}((s_u)_{u \in W_v}; \vec{s}')$ is independent of the choice of $(s'_u)_{u \in V_v^\circ}$, we find that, for fixed $(s_u)_{u \in W_v}$,

$$(4.3) \quad (\Psi_t|_{T_v})_{t \in [a, 1]} \text{ defines a } (\gamma_v, T_v)\text{-adapted deformation of } \mathcal{D}(\gamma_v(a)).$$

Proof of Proposition 4.3. We prove the following for every $v \in V$ by the induction used in the proof of Proposition 3.8:

$$(4.4) \quad (\mathfrak{p}_{\Omega_v}^* \hat{\psi}_{T_v})(\underline{\gamma}_v(t)) = (\Psi_t|_{T_v})_\# [\Delta_{T_v}](\beta_{T_v})$$

holds for every $t \in [a, 1]$ and $(s_u)_{u \in W_v}$ satisfying the condition (4.2). For the case $v \in L$, the equation (4.4) follows from the definition of γ_v .

Now, assume that the equation (4.4) holds for all the vertexes $u \in V_v^\circ$. Let \tilde{T}_v be an iteration diagram defined by the vertexes $\tilde{V}_v := V_v^1 \cup \{v\}$ and the edges of the form $u \rightarrow v$ ($u \in V_v^1$). We set $\Psi_t|_{\tilde{T}_v} := \text{pr}_{\tilde{T}_v} \circ \Psi_t|_{T_v}$, where $\text{pr}_{\tilde{T}_v} : X_\Omega^{T_v} \rightarrow X_\Omega^{\tilde{T}_v} := \prod_{u \in \tilde{V}_v} X_{\Omega_u}$

is the natural projection. Since $\Psi_t|_{\tilde{T}_v}((s_u)_{u \in W_v}; \vec{s'})$ depends only on the variables s_u ($u \in \tilde{V}_v$), it defines a map $\Psi_t|_{\tilde{T}_v} : \Delta_{\tilde{T}_v} \rightarrow X_{\Omega}^{\tilde{T}_v}$. We then obtain

$$(\Psi_t|_{T_v})_{\#}[\Delta_{T_v}](\beta_{T_v}) = (\Psi_t|_{\tilde{T}_v})_{\#}[\Delta_{\tilde{T}_v}](\tilde{\beta}_{\tilde{T}_v})$$

from the induction hypothesis, where $\tilde{\beta}_{\tilde{T}_v}$ is a holomorphic 1-form defined by

$$\tilde{\beta}_{\tilde{T}_v} := (\mathfrak{p}_{\Omega_v}^* \hat{\varphi}_v)(\underline{\xi}_v) \hat{f}_v(\xi_v - \sum_{u \in V_v^1} \xi_u) \prod_{u \in V_v^1} (\mathfrak{p}_{\Omega_u}^* \hat{\psi}_{T_u})(\underline{\xi}_u) \bigwedge_{u \in V_v^1} d\underline{\xi}_u.$$

Applying Lemma 4.4, we have

$$\begin{aligned} (\Psi_t|_{\tilde{T}_v})_{\#}[\Delta_{\tilde{T}_v}](\tilde{\beta}_{\tilde{T}_v}) &= \mathfrak{p}_{\Omega_v}^* \left(\hat{\varphi}_v \cdot \left(\hat{f}_v * \prod_{u \in V_v^1}^* \hat{\psi}_{T_u} \right) \right) (\underline{\gamma}_v(t)) \\ &= (\mathfrak{p}_{\Omega_v}^* \hat{\psi}_{T_v})(\underline{\gamma}_v(t)). \end{aligned}$$

It proves (4.4) for all $v \in V$. Since $\gamma_{\hat{v}} = \gamma$ and $T_{\hat{v}} = T$, we obtain Proposition 4.3. \square

The proof of Theorem 3.9 is reduced to the following

Theorem 4.6. *Let $T = (V, E)$ be an iteration diagram in \mathcal{T}_k ($k \geq 1$) and assume that $\gamma \in \Pi_{\Omega_{\hat{v}}}^{\delta, L}$ for $L > 0$ and $\delta \in (0, \rho]$ satisfying the condition (4.1) is given. Then, there exists a (γ, T) -adapted deformation $(\Psi_t)_{t \in [a, 1]}$ of $\mathcal{D}(\gamma(a))$ such that*

$$(4.5) \quad \Psi_t(\Delta_T) \subset \bigcup_{\vec{s} \in \Delta_T} \prod_{v \in V} K_{\Omega_v}^{\delta'(t), s_v L(\gamma|_t)}.$$

Further, the partial derivatives $\partial \xi_v^t / \partial s_u$ are defined almost everywhere on Δ_T and satisfy

$$(4.6) \quad \left| \det \left[\frac{\partial \xi_v^t}{\partial s_u} \right]_{u, v \in V^{\circ}} \right| \leq (c(t))^{k-1},$$

where

$$(4.7) \quad \delta'(t) := \rho e^{-2\sqrt{2}\delta^{-1}L_a(\gamma|_t)}, \quad c(t) := \rho e^{3\delta^{-1}L_a(\gamma|_t)},$$

$$L_a(\gamma|_t) = \int_a^t |\gamma'(t')| dt'.$$

See [KaS] for the reduction of Theorem 3.9 to Theorem 4.6. The proof of Theorem 4.6 will be given in Section 5 and 6.

§ 5. Construction of a (γ, T) -adapted deformation

In this section, we construct a (γ, T) -adapted deformation of $\mathcal{D}(\gamma(a))$ satisfying the conditions in Theorem 4.6 by the method developed in [S3], [S4], [OD] and [KaS]. Let us assume that $T = (V, E) \in \mathcal{T}_k$, a d.f.s. Ω and $\gamma \in \Pi_{\Omega_{\hat{v}}}^{\delta, L}$ satisfying the assumptions in Theorem 4.6 are given.

Notation 5.1. We define functions η_v and D_v ($v \in V$) by

$$\begin{aligned} \zeta = (\lambda, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{C} &\mapsto \eta_v(\zeta) := \text{dist}((\lambda, \xi), \{(0, 0)\} \cup \overline{\mathcal{S}_{\Omega_v}}), \\ (\zeta_v, (\zeta_u)_{u \in V_v^1}) \in (\mathbb{R}_{\geq 0} \times \mathbb{C})^{|V_v^1|+1} &\mapsto D_v(\zeta_v, (\zeta_u)_{u \in V_v^1}) := \sum_{u \in V_v^1} \eta_u(\zeta_u) + \left| \zeta_v - \sum_{u \in V_v^1} \zeta_u \right|, \end{aligned}$$

where $\text{dist}(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$.

By the definition of η_v , we find that η_v is Lipschitz continuous on $\mathbb{R}_{\geq 0} \times \mathbb{C}$. More precisely, we have the following:

$$|\eta_v(\zeta) - \eta_v(\zeta')| \leq |\zeta - \zeta'| \quad \text{holds for every } \zeta, \zeta' \in \mathbb{R}_{\geq 0} \times \mathbb{C}.$$

We then see that the following holds for every ζ_v, ζ'_v and ζ_u, ζ'_u ($u \in V_v^1$) in $\mathbb{R}_{\geq 0} \times \mathbb{C}$:

$$|D_v(\zeta_v, (\zeta_u)_{u \in V_v^1}) - D_v(\zeta'_v, (\zeta'_u)_{u \in V_v^1})| \leq |\zeta_v - \zeta'_v| + 2 \sum_{u \in V_v^1} |\zeta_u - \zeta'_u|.$$

Lemma 5.2. *The functions η_v and D_v satisfy the following inequality for every ζ_v and ζ_u ($u \in V_v^1$) in $\mathbb{R}_{\geq 0} \times \mathbb{C}$:*

$$(5.1) \quad D_v(\zeta_v, (\zeta_u)_{u \in V_v^1}) \geq \eta_v(\zeta_v).$$

Proof. For each $\zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C}$ ($u \in V$), we can take $\tilde{\zeta}_u \in \overline{\mathcal{S}_{\Omega_u}} \cup \{(0, 0)\}$ so that $\eta_u(\zeta_u) = |\zeta_u - \tilde{\zeta}_u|$ holds. Therefore, we find the following inequality:

$$D_v(\zeta_v, (\zeta_u)_{u \in V_v^1}) = \sum_{u \in V_v^1} |\zeta_u - \tilde{\zeta}_u| + \left| \zeta_v - \sum_{u \in V_v^1} \zeta_u \right| \geq \left| \zeta_v - \sum_{u \in V_v^1} \tilde{\zeta}_u \right|.$$

Since $\sum_{u \in V_v^1} \tilde{\zeta}_u \in \overline{\mathcal{S}_{\Omega_v}} \cup \{(0, 0)\}$, we obtain (5.1). □

We now define a family of functions

$$\zeta_v^t : \Delta_T \rightarrow \mathcal{M}_{\Omega_v} \quad (t \in [a, 1], v \in V)$$

by the following process: We first define ζ_v^t by $\zeta_v^t(\vec{s}) := \tilde{\gamma}(t)$ for all $\vec{s} \in \Delta_T$. Next, assume that $\zeta_v^t = (\lambda_v^t, \xi_v^t)$ has already determined and that

$$(5.2) \quad \zeta_v^t(\vec{s}) \text{ is } C^1 \text{ on } [a, 1] \text{ and } \lambda_v^t(\vec{s}) \text{ is increasing for each } \vec{s} \in \Delta_T,$$

$$(5.3) \quad \lambda_v^a(\vec{s}) = 0 \text{ if and only if } s_v = 0.$$

Then, we define ζ_u^t ($u \in V_v^1$) as follows:

- When $s_v = 0$, we set $\zeta_u^t(\vec{s}) := 0$ for $t \in [a, 1]$.
- When $s_v > 0$, we define $\zeta_u^t(\vec{s})$ ($u \in V_v^1$) by the solution of the differential equation

$$(5.4.u) \quad \frac{d\zeta_u}{dt} = \frac{\eta_u(\zeta_u)}{D_v(\zeta_v^t, (\zeta_u)_{u \in V_v^1})} \frac{\partial \zeta_v^t}{\partial t}$$

with the initial condition $\zeta_u|_{t=a} = s_u \tilde{\gamma}(a)$.

Notice that, by the induction hypotheses and Lemma 5.2, we can take $\varepsilon > 0$ so that $D_v(\zeta_v^t, (\zeta_u)_{u \in V_v^1}) \geq \varepsilon$ holds for every $t \in [a, 1]$ and $\zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C}$ ($u \in V_v^1$) when $s_v > 0$. We then see that the right hand side of (5.4.u) is locally Lipschitz continuous on the variables ζ_u ($u \in V_v^1$). Therefore, since $s_u \tilde{\gamma}(a) \notin \{\zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \eta_u(\zeta_u) = 0\} = \overline{\mathcal{S}}_{\Omega_u} \cup \{(0, 0)\}$ when $s_u \neq 0$, the Cauchy-Lipschitz theorem yields the existence and uniqueness of the solutions $\zeta_u : [a, 1] \rightarrow \mathcal{M}_{\Omega_u} \setminus \{(0, 0)\}$ ($u \in V_u^1$) of (5.4.u) satisfying the initial condition $\zeta_u|_{t=a} = s_u \tilde{\gamma}(a)$. We immediately derive from the construction of ζ_u^t ($u \in V_v^1$) that they also satisfy the induction hypotheses (5.2) and (5.3). In such a way, we can inductively determine a family of functions ζ_v^t ($v \in V$).

We now define a family of maps

$$\Phi_t : \Delta_T \rightarrow \mathcal{M}_{\Omega}^T := \prod_{v \in V} \mathcal{M}_{\Omega_v} \quad (t \in [a, 1])$$

by $\Phi_t := (\zeta_v^t)_{v \in V}$. Since each $\zeta_v^t(\vec{s})$ satisfies

$$\lambda_v^t(\vec{s}) = s_v |\gamma(a)| + \int_a^t \left| \frac{\partial \zeta_v^{t'}}{\partial t'}(\vec{s}) \right| dt',$$

we find that the path $\tilde{\gamma}_v(\vec{s}) : [0, 1] \rightarrow \mathcal{M}_{\Omega_v}$ obtained by concatenating the paths $s_v \tilde{\gamma}|_{[0, a]}$ and $\zeta_v(\vec{s}) : [a, 1] \rightarrow \mathcal{M}_{\Omega_v}$ defines an Ω_v -allowed path $\gamma_v(\vec{s})$ and its lift $\underline{\gamma}_v(\vec{s})$ on X_{Ω_v} . We then define a family of maps $\Psi_t : \Delta_T \rightarrow X_{\Omega}^T$ ($t \in [a, 1]$) by $\Psi_t(\vec{s}) := (\underline{\gamma}_v^t(\vec{s}))_{v \in V}$.

We now confirm that $(\Psi_t)_{t \in [a, 1]}$ gives a (γ, T) -adapted deformation of $\mathcal{D}(\gamma(a))$. It suffices to show the following properties of $(\Phi_t)_{t \in [a, 1]}$:

Lemma 5.3. *For each $v \in V$ and $t \in [a, 1]$, the function ζ_v^t on Δ_T only depends on the variables s_u ($u \in W_v$).*

Lemma 5.4. *For every $t \in [a, 1]$, the followings hold:*

$$(5.5) \quad \zeta_v^t = 0 \quad \text{when} \quad s_v = 0 \quad (v \in V^\circ),$$

$$(5.6) \quad \sum_{u \in V_v^1} \zeta_u^t = \zeta_v^t \quad \text{when} \quad \sum_{u \in V_v^1} s_u = s_v \quad (v \in V \setminus L).$$

Proposition 5.5. *The map $\Phi : [a, 1] \times \Delta_T \rightarrow \mathcal{M}_\Omega^T$ is Lipschitz continuous.*

Lemma 5.3 immediately follows from the construction of ζ_v^t .

Proof of Lemma 5.4. Since $\eta_v(0) = 0$ for $v \in V^\circ$, we find that the hyperplane defined by $\zeta_v = 0$ is invariant by the flow of (5.4.v) when $s_{v_\uparrow} > 0$. Therefore, combining the case $s_{v_\uparrow} = 0$, we obtain (5.5). We then prove (5.6). We first consider the case $s_v > 0$. Since ζ_u^t ($u \in V_v^1$) satisfies (5.4.u), we find that $\Xi := \zeta_v^t - \sum_{u \in V_v^1} \zeta_u^t$ satisfies

$$(5.7) \quad \frac{d\Xi}{dt} := \frac{|\Xi|}{D_v((\zeta_u^t)_{u \in V_v^1}, \Xi)} \frac{\partial \zeta_v^t}{\partial t}$$

and an initial condition $\Xi|_{t=a} = (s_v - \sum_{u \in V_v^1} s_u) \tilde{\gamma}(a)$, where

$$D_v((\zeta_u)_{u \in V_v^1}, \Xi) := \sum_{u \in V_v^1} \eta_u(\zeta_u) + |\Xi|.$$

Therefore, since $\Xi = 0$ also satisfies (5.7) and the initial condition when $\sum_{u \in V_v^1} s_u = s_v$, we obtain (5.6) from the uniqueness of such solutions. On the other hand, the equation (5.6) is obvious when $s_v = 0$ because $\zeta_v^t = 0$ and $\zeta_u^t = 0$ ($u \in V_v^1$). \square

Remark 5.6. Since the inequality

$$\sum_{u \in V_v^1} \eta_u(\zeta_u) \leq D_v(\zeta_v, (\zeta_u)_{u \in V_v^1})$$

holds for $\zeta_v, \zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C}$, we can derive from (5.4.u) the following inequality:

$$(5.8) \quad \sum_{u \in V_v^1} \lambda_u^t \leq \lambda_v^t \quad \text{holds for } t \in [a, 1].$$

For the proof of Proposition 5.5, we use the following

Lemma 5.7. *If λ_v^t ($v \in V \setminus L$) satisfies $0 < \lambda_v^t(\vec{s}) \leq \rho$ at $\vec{s} \in \Delta_T$ and $t \in [a, 1]$, $\zeta_u^t(\vec{s})$ ($u \in V_v^1$) is written as $\zeta_u^t(\vec{s}) = s_u s_v^{-1} \zeta_v^t(\vec{s})$.*

Proof. Since λ_u^t ($u \in V_v^1$) satisfies $\lambda_u^{t'}(\vec{s}) \leq \lambda_v^t(\vec{s}) \leq \rho$ for $t' \in [a, t]$, we find that $\eta_u(\zeta_u^{t'}(\vec{s})) = |\zeta_u^{t'}(\vec{s})|$ holds by the definition of η_u . Therefore, since $\zeta_u = s_u s_v^{-1} \zeta_v^t$ also gives a solution of (5.4.u) satisfying the initial condition $\zeta_u|_{t=a} = s_u \tilde{\gamma}(a)$, we obtain the representation of ζ_u^t from the uniqueness of such solutions. \square

Proof of Proposition 5.5. We prove by induction that each of the functions ζ_v^t ($v \in V$) is Lipschitz continuous on Δ_T . Since $\zeta_{\hat{v}}^t$ is independent of $\vec{s} \in \Delta_T$, the statement holds for $v = \hat{v}$. We then assume that λ_v^t is Lipschitz continuous on Δ_T . Since the right

hand side of (5.4.u) ($u \in V_v^1$) is Lipschitz continuous on the variables $s_{v'}$ ($v' \in W_v$) by the induction hypothesis, we find by the use of the Cauchy-Lipschitz theorem that ζ_u^t is locally Lipschitz continuous on the variables $s_{v'}$ ($v' \in W_u$) when $s_v > 0$. Further, we see by the use of the representation of ζ_u^t in Lemma 5.7 that the following holds for $\vec{s}, \vec{s}' \in \Delta_T$ with s_v, s'_v sufficiently small:

$$|\zeta_u^t(\vec{s}) - \zeta_u^t(\vec{s}')| \leq \frac{|\zeta_v^t(\vec{s})|}{s_v} |s_u - s'_u| + \frac{s'_u}{s_v} |\zeta_v^t(\vec{s}) - \zeta_v^t(\vec{s}')| + \frac{s'_u}{s_v} \frac{|\zeta_v^t(\vec{s}')|}{s'_v} |s'_v - s_v|.$$

We may assume without loss of generality that $s'_v \leq s_v$, and hence, $s'_u \leq s_v$. Therefore, we find by the induction hypothesis that we can take the Lipschitz constant of ζ_u^t uniformly in a neighborhood of $s_v = 0$. Thus, we have the Lipschitz continuity of ζ_u^t on Δ_T . Since $\zeta_u^t(\vec{s})$ is C^1 on $[a, 1]$ for each $\vec{s} \in \Delta_T$, we obtain the Lipschitz continuity of the map Φ . \square

§ 6. Estimates for the (γ, T) -adapted deformation

In the previous section, we constructed the map $\Phi : [a, 1] \times \Delta_T \rightarrow \mathcal{M}_\Omega^T$ for $\gamma \in \Pi_{\Omega_{\hat{v}}}^{\delta, L}$ satisfying the assumptions in Theorem 4.6 and the properties of the (γ, T) -adapted deformation $(\Psi_t)_{t \in [a, 1]}$ was reduced to its properties. In this section, we derive estimates for $\Phi = (\zeta_v)_{v \in V}$ which are necessary to prove Theorem 4.6. We first show the following

Lemma 6.1. *For every $v \in V$, $t \in [a, 1]$ and $\vec{s} \in \Delta_T$, the function $\zeta_v^t(\vec{s})$ satisfies*

$$(6.1) \quad \text{dist}(\zeta_v^t(\vec{s}), \overline{\mathcal{S}}_{\Omega_v}) \geq \delta'(t),$$

where $\delta'(t)$ is the function given in (4.7).

Proof. When $s_v = 0$, (6.1) immediately follows from (5.5). We then assume that $s_v > 0$. Let $\{v_j\}_{j=1}^\ell$ be vertexes such that $v = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_\ell = \hat{v}$ gives a path on T from v to \hat{v} . Since $s_{v_j} \geq s_v$ holds for $j = 1, \dots, \ell$, we find that $\zeta_j^t := \zeta_{v_j}^t$ satisfies $\eta_j(\zeta_j^t(\vec{s})) > 0$ for every $t \in [a, 1]$, where $\eta_j := \eta_{v_j}$. We now consider estimates of the function $h_j^t := \eta_j(\zeta_j^t(\vec{s}))$. Since h_1^t is Lipschitz continuous on $[a, 1]$, we find by Rademacher's theorem that the derivative of h_1^t exists a.e. on $[a, 1]$ and satisfies $|dh_1^t/dt| \leq |\partial \zeta_1^t / \partial t|$. We further obtain from Lemma 5.2 and (5.4.v_j) the following sequence of inequalities:

$$(6.2) \quad \frac{1}{h_1^t} \left| \frac{\partial \zeta_1^t}{\partial t} \right| \leq \frac{1}{h_2^t} \left| \frac{\partial \zeta_2^t}{\partial t} \right| \leq \cdots \leq \frac{1}{h_\ell^t} \left| \frac{\partial \zeta_\ell^t}{\partial t} \right| \leq \frac{|\tilde{\gamma}'|}{\delta}.$$

Thus, we have the following:

$$(6.3) \quad \frac{1}{h_1^t} \left| \frac{dh_1^t}{dt} \right| \leq \frac{|\tilde{\gamma}'|}{\delta} \quad \text{holds a.e. on } [a, 1].$$

Since $h_1^a = \sqrt{2}s_v\rho$ and $|\tilde{\gamma}'| = \sqrt{2}|\gamma'|$, we derive from (6.3) the following:

$$\sqrt{2}s_v\rho e^{-\sqrt{2}\delta^{-1}L_a(\gamma|_t)} \leq h_1^t \leq \sqrt{2}s_v\rho e^{\sqrt{2}\delta^{-1}L_a(\gamma|_t)}.$$

Therefore, we find that $\rho e^{-2\sqrt{2}\delta^{-1}L_a(\gamma|_t)} \leq h_1^t$ holds for any $t \in [a, 1]$ when $\sqrt{2}s_v \geq e^{-\sqrt{2}\delta^{-1}L_a(\gamma|_t)}$. Otherwise, we have $h_1^t \leq \rho$, and hence, $|\zeta_1^t| \leq \rho$ holds for any $t \in [a, 1]$. It proves (6.1). \square

We next show that $\Phi_t = ((\lambda_v^t, \xi_v^t))_{v \in V}$ satisfies the estimate (4.6). We use the following

Lemma 6.2. *Suppose that $\vec{s}, \vec{s}' \in \Delta_T$ satisfy $s_u = s'_u$ for $u \in V \setminus V_v^1$ with $v \in V \setminus L$. Then, we have for every $t \in [a, 1]$*

$$(6.4) \quad \sum_{u \in V_v^1} \left| \frac{\partial \xi_u^t}{\partial t}(\vec{s}) - \frac{\partial \xi_u^t}{\partial t}(\vec{s}') \right| \leq 3 \frac{|\gamma'|}{\delta} \sum_{u \in V_v^1} |\xi_u^t(\vec{s}) - \xi_u^t(\vec{s}')|.$$

Proof. Since the left hand side of (5.4.u) satisfies

$$\begin{aligned} & \sum_{u' \in V_v^1} \left| \left(\frac{\eta_{u'}(\zeta_{u'})}{D_v(\zeta_v^t, (\zeta_u)_{u \in V_v^1})} - \frac{\eta_{u'}(\zeta'_{u'})}{D_v(\zeta_v^t, (\zeta'_u)_{u \in V_v^1})} \right) \frac{\partial \xi_v^t}{\partial t} \right| \\ & \leq \frac{|D_v(\zeta_v^t, (\zeta'_u)_{u \in V_v^1}) - D_v(\zeta_v^t, (\zeta_u)_{u \in V_v^1})|}{D_v(\zeta_v^t, (\zeta'_u)_{u \in V_v^1})} \left| \frac{\partial \xi_v^t}{\partial t} \right| \sum_{u' \in V_v^1} \frac{\eta_{u'}(\zeta_{u'})}{D_v(\zeta_v^t, (\zeta_u)_{u \in V_v^1})} \\ & \quad + \left| \frac{\partial \xi_v^t}{\partial t} \right| \sum_{u' \in V_v^1} \frac{|\eta_{u'}(\zeta_{u'}) - \eta_{u'}(\zeta'_{u'})|}{D_v(\zeta_v^t, (\zeta'_u)_{u \in V_v^1})} \\ & \leq \frac{3}{D_v(\zeta_v^t, (\zeta'_u)_{u \in V_v^1})} \left| \frac{\partial \xi_v^t}{\partial t} \right| \sum_{u' \in V_v^1} |\zeta_{u'} - \zeta'_{u'}|, \end{aligned}$$

we obtain (6.4) by the use of (6.2) when $s_v > 0$. The inequality (6.4) is trivial when $s_v = 0$. \square

We now set for $\vec{s}, \vec{s}' \in \Delta_T$ and $v \in V \setminus L$

$$\Xi_v(t) := \sum_{u \in V_v^1} |\xi_u^t(\vec{s}) - \xi_u^t(\vec{s}')|.$$

Under the assumption in Lemma 6.2, we obtain from (6.4) the following:

$$|\Xi_v(t) - \Xi_v(a)| \leq \frac{3}{\delta} \int_a^t \Xi_v(t') |\gamma'(t')| dt'.$$

Therefore, Gronwall's Lemma yields the following estimate:

$$(6.5) \quad \Xi_v(t) \leq c(t) \Xi_v(a) \quad \text{holds for every } t \in [a, 1],$$

where $c(t)$ is the constant given in (4.7). Since $\Xi_v(a) = \rho \sum_{u \in V_v^1} |s_u - s'_u|$, we see by (6.5) that the following holds a.e. on Δ_T :

$$(6.6) \quad \left| \det \left[\frac{\partial \xi_{u'}}{\partial s_u} \right]_{u, u' \in V_v^1} \right| \leq \prod_{u \in V_v^1} \left(\sum_{u' \in V_v^1} \left| \frac{\partial \xi_{u'}}{\partial s_u} \right| \right) \leq (c(t))^{|V_v^1|}.$$

Notice that $\partial \xi_v / \partial s_u = 0$ for $u \notin W_v$, and hence, we find the following:

$$(6.7) \quad \left| \det \left[\frac{\partial \xi_{u'}}{\partial s_u} \right]_{u, u' \in V^\circ} \right| = \prod_{v \in V \setminus L} \left| \det \left[\frac{\partial \xi_{u'}}{\partial s_u} \right]_{u, u' \in V_v^1} \right|.$$

Since $\sum_{v \in V \setminus L} |V_v^1| = k - 1$, we obtain (4.6) from (6.6) and (6.7). Together with (5.8) and Lemma 6.1, Theorem 4.6 is validated.

§ 7. Applications to nonlinear differential and difference equations

In this section, we discuss the resurgence of formal series solutions

$$\Phi = \begin{pmatrix} \Phi^{(1)} \\ \vdots \\ \Phi^{(n)} \end{pmatrix} \in \mathbb{C}^n[[x^{-1}]]$$

of a nonlinear differential equation

$$(7.1) \quad \frac{d}{dx} \Phi = F(x^{-1}, \Phi)$$

at $x = \infty$ with $F(x^{-1}, \Phi) \in \mathbb{C}^n\{x^{-1}, \Phi\}$ satisfying the conditions

$$(7.2) \quad F(0, 0) = 0 \quad \text{and} \quad \det(\partial_\Phi F(0, 0)) \neq 0.$$

Under the assumption (7.2), there exists a unique formal series solution of the form

$$\Phi(x) = \sum_{k=1}^{\infty} \Phi_k x^{-k}.$$

We rewrite $F(x^{-1}, \Phi)$ in the following form:

$$F(x^{-1}, \Phi) = F_0(x^{-1}) + \partial_\Phi F(0, 0)\Phi + \sum_{|\ell| \geq 1} F_\ell(x^{-1})\Phi^\ell,$$

where $\Phi^\ell := (\Phi^{(1)})^{\ell_1} \cdots (\Phi^{(n)})^{\ell_n}$ with $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}_{\geq 0}^n$ and $|\ell| := \ell_1 + \cdots + \ell_n$. Regarding (7.1) as an equation for $\tilde{\Phi}(x) := x^{-1}(\Phi(x) - \Phi_1 x^{-1})$, we may assume without loss of generality that

$$(7.3) \quad F_\ell(0) = 0 \quad \text{for every} \quad \ell \in \mathbb{Z}_{\geq 0}^n.$$

Applying the Borel transform, (7.1) is rewritten as follows:

$$(7.4) \quad P(\xi)\hat{\Phi} = \hat{F}_0 + \sum_{|\ell| \geq 1} \hat{F}_\ell * \hat{\Phi}^{*\ell},$$

where $P(\xi) := -\xi - \partial_\Phi F(0, 0)$ and $\hat{\Phi}^{*\ell} := \mathcal{B}(\Phi^\ell)$. We now inductively determine $\hat{\Phi}_k$ ($k \geq 1$) by the following procedure:

$$(7.5) \quad \hat{\Phi}_1 := P^{-1}\hat{F}_0,$$

$$(7.6) \quad \hat{\Phi}_{k+1} := P^{-1} \sum_{j=1}^k \sum_{|\ell|=j} \hat{F}_\ell * \sum_{\substack{k_1+\dots+k_j=k \\ k_i \geq 1}} \hat{\Phi}_{k_1, \dots, k_j}^\ell,$$

where $\hat{\Phi}_{k_1, \dots, k_j}^\ell$ is the convolution product of functions in

$$\left\{ \hat{\Phi}_{k_i}^{(m)} \mid 1 + \sum_{p=1}^{m-1} \ell_p \leq i \leq \sum_{p=1}^m \ell_p, 1 \leq m \leq n \right\}.$$

We now introduce the following

Definition 7.1. Let $T = (V, E) \in \mathcal{T}$ and consider a function $\nu = (\nu_1, \nu_2) : V \rightarrow \{1, \dots, n\}^2$. We call such a pair $\overline{T} = (T, \nu)$ *n-decorated iteration diagram* and denote the set of *n*-decorated iteration diagrams by \mathcal{T}^n .

Let $\overline{T} = (T, \nu) \in \mathcal{T}^n$. We define an equivalence relation \sim_v on V_v^1 for $v \in V \setminus L$ as follows: $u \sim_v u'$ ($u, u' \in V_v^1$) if $T_u = T_{u'}$ and $\nu|_{V_u} = \nu|_{V_{u'}}$. For each $[u] \in V_v^1 / \sim_v$, we define an integer $\#[u]$ as the cardinal of $\{u' \in V_v^1 \mid u' \sim_v u\}$ and the multiplicity μ_v of \overline{T} at v by

$$\mu_v := \left(\prod_{[u] \in V_v^1 / \sim_v} (\#[u])! \right)^{-1} \cdot (|V_v^1|)!.$$

We set $\mu_{\overline{T}} := \prod_{v \in V \setminus L} \mu_v$. We further define a map $\lambda_{\overline{T}, j} : V \rightarrow \mathbb{Z}_{\geq 0}$ ($j = 1, \dots, n$) by $\lambda_{\overline{T}, j}(v) := |\{u \in V_v^1 \mid \nu_1(u) = j\}|$ when $v \in V \setminus L$ and $\lambda_{\overline{T}, j}(v) := 0$ when $v \in L$. We set $\lambda_{\overline{T}} = (\lambda_{\overline{T}, 1}, \dots, \lambda_{\overline{T}, n})$. By the use of the multinomial theorem, we obtain the following

Lemma 7.2. For every $k \geq 1$ and $j \in \{1, \dots, n\}$, $\hat{\Phi}_k^{(j)}$ is written as follows:

$$(7.7) \quad \hat{\Phi}_k^{(j)} = \sum_{\overline{T} \in \mathcal{T}_k^{n, (j)}} \mu_{\overline{T}} \cdot \psi_{\overline{T}},$$

where $\mathcal{T}_k^{n, (j)} := \{(T, \nu) \in \overline{\mathcal{T}} \mid T \in \mathcal{T}_k, \nu_1(\hat{v}) = j\}$ and $\psi_{\overline{T}}$ is the iterated convolution product of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ with $\hat{f}_v := \hat{F}_{\lambda_{\overline{T}}(v)}^{(\nu_2(v))}$ and $\hat{\varphi}_v := (P^{-1})_{\nu(v)}((\nu_1(v), \nu_2(v))\text{-th entry of } P^{-1})$.

We now show the following

Lemma 7.3. *There exist positive constants C and $\delta(< 1/n)$ such that*

$$(7.8) \quad \sum_{\overline{T} \in \mathcal{T}_k^{n,(j)}} \mu_{\overline{T}} \leq \delta B(k) C^k$$

holds for ever $k \geq 1$ and $j \in \{1, \dots, n\}$, where $B(k)$ is a constant defined by

$$B(k) := \frac{3}{2\pi^2(k+1)^2}.$$

Proof. Since the left hand side of (7.8) is independent of j , we denote it by N_k . We find by (7.6) that $\{N_k\}_{k \geq 1}$ satisfy the following:

$$(7.9) \quad N_{k+1} = \sum_{j=1}^k \sum_{|\ell|=j} n \sum_{\substack{k_1+\dots+k_j=k \\ k_i \geq 1}} N_{k_1} \cdots N_{k_j}.$$

Taking $C > 0$ sufficiently large, we may assume that N_1 satisfies (7.8) for arbitrary small positive δ . We then assume that (7.8) holds for $k \leq K$. Using the inequality

$$\sum_{k_1+\dots+k_j=k} B(k_1) \cdots B(k_j) \leq B(k),$$

we derive from (7.9) the following:

$$N_{K+1} \leq nB(K)C^K \sum_{j=1}^K \sum_{|\ell|=j} \delta^j \leq nB(K)C^K \sum_{j=1}^K (n\delta)^j \leq nB(K)C^K \frac{n\delta}{1-n\delta}.$$

Therefore, taking C sufficiently large so that it satisfies $n^2(1-n\delta)^{-1} \leq C$, we see that (7.8) holds for $k = K+1$. We thus obtain (7.8) for $k \geq 1$. \square

Let $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ be a d.f.s. defined by the formula

$$(7.10) \quad \Omega_L = \{\xi \in \mathbb{C} \mid \det(-\xi - \partial_\Phi F(0, 0)) = 0, |\xi| \leq L\}.$$

We then find that each entry of P^{-1} is Ω -continuable, and hence, we obtain from Corollary 3.10 the following estimates: for any $\delta, L > 0$, there exist $c, \delta' > 0$ such that

$$(7.11) \quad \|\hat{\psi}_{\overline{T}}\|_{\Omega_{\hat{v}}}^{\delta, L} \leq \frac{c^{k-1}}{(k-1)!} \prod_{v \in V} \|(P^{-1})_{\nu(v)}\|_{\Omega}^{\delta', L} \|\hat{F}_{\lambda_{\overline{T}}(v)}^{(\nu_2(v))}\|_{\emptyset}^{\delta', L}$$

holds for every $\overline{T} \in \mathcal{T}_k^{n,(j)}$. Since $F(x^{-1}, \Phi) \in \mathbb{C}^n\{x^{-1}, \Phi\}$, we can take $A > 0$ so that $\|(P^{-1})_{ij}\|_{\Omega}^{\delta', L} \leq A$ and $\|\hat{F}_{\ell}^{(j)}\|_{\emptyset}^{\delta', L} \leq A^{1+|\ell|}$ hold for any $i, j \in \{1, \dots, n\}$ and $\ell \in \mathbb{Z}_{\geq 0}^n$.

Notice that $|\lambda_{\overline{T}}(v)| = |V_v^1|$ when $v \in V \setminus L$, and hence, $\sum_{v \in V} |\lambda_{\overline{T}}(v)| = k-1$. Therefore, we derive from (7.7), (7.8) and (7.11) the following estimates: there exists a positive constant C such that

$$(7.12) \quad \|\hat{\Phi}_k^{(j)}\|_{\Omega^{*\infty}}^{\delta, L} \leq \frac{C^k}{(k-1)!}$$

holds for every $k \geq 1$ and $j \in \{1, \dots, n\}$. We then find that each entry of $\hat{\Phi} = \sum_{k \geq 1} \hat{\Phi}_k$ converges in $\hat{\mathcal{R}}_{\Omega^{*\infty}}$ and $\hat{\Phi}$ gives a solution of (7.4). Thus, we obtain the following

Theorem 7.4. *Let $\Omega = \{\Omega_L\}_{L \in \mathbb{R}_{\geq 0}}$ be a d.f.s. defined by (7.10). Then, each entry of the formal series solution $\Phi \in \mathbb{C}^n[[x^{-1}]]$ of (7.1) is $\Omega^{*\infty}$ -resurgent.*

By the same discussion, we have the following

Theorem 7.5. *Let us consider a nonlinear difference equation*

$$(7.13) \quad \Phi(x+1) - \Phi(x) = F(x^{-1}, \Phi(x))$$

at $x = \infty$ under the assumption (7.2). Then, there exists a unique formal series solution $\Phi(x) \in \mathbb{C}^n[[x^{-1}]]$ of (7.13) and each entry of $\Phi(x)$ is $\Omega^{*\infty}$ -resurgent, where $\Omega = \{\Omega_L\}_{L \in \mathbb{R}_{\geq 0}}$ is a d.f.s. defined by

$$(7.14) \quad \Omega_L = \{\xi \in \mathbb{C} \mid \det((e^{-\xi} - 1) - \partial_{\Phi} F(0, 0)) = 0, |\xi| \leq L\}.$$

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